The second step is for curves that cross themselves, like the one in Figure 16.68b. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes' Theorem one loop at a time, and add the results. ■

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.

	Theorem 2, Section 16.3	
F conservative on D	$\Longleftrightarrow$	$\mathbf{F} = \nabla f$ on D
Theorem 3, Section 16.3 $\Updownarrow$		$\Downarrow$ Vector identity (Eq. 8) (continuous second partial derivatives)
$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ over any closed path in D	$\Longleftarrow$ Theorem 7 Domain's simple connectivity and Stokes' Theorem	$\nabla \times \mathbf{F} = \mathbf{0}$ throughout D

Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field **F** around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$
   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$
   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above

3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above

6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$
   C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$, counterclockwise when viewed from above

Integral of the Curl Vector Field

7. Let **n** be the outer unit normal of the elliptical shell

$$S:\quad 4x^2 + 9y^2 + 36z^2 = 36, \qquad z \ge 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2}\sin e^{\sqrt{xyz}}\,\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$.)

8. Let **n** be the outer unit normal (normal away from the origin) of the parabolic shell

$$S:\quad 4x^2 + y + z^2 = 4, \qquad y \ge 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2 + x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4 + z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with its top, $x^2 + y^2 \le a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S.

10. Evaluate

$$\iint_S \nabla \times (y\mathbf{i})\cdot\mathbf{n}\,d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$.

ɔpose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{xyz}\mathbf{j} + \cos(xz)\mathbf{k}.$$

termine the flux of $\mathbf{F}$ outward through the hemisphere $+ y^2 + z^2 = 1, z \geq 0$.

ɒeat Exercise 11 for the flux of $\mathbf{F}$ across the entire unit sphere.

Theorem for Parametrized Surfaces

cises 13–18, use the surface integral in Stokes' Theorem to e the flux of the curl of the field $\mathbf{F}$ across the surface S in the n of the outward unit normal $\mathbf{n}$.

$= 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$

$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4 - r^2)\mathbf{k}$,
$\leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$

$= (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$

$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$,
$\leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$

$= x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$,
$\leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$

$= (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$,
$\leq r \leq 5, \quad 0 \leq \theta \leq 2\pi$

$= 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$

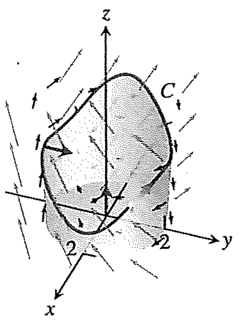
$\mathbf{r}(\phi, \theta) = (\sqrt{3}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{3}\sin\phi\sin\theta)\mathbf{j} +$
$\sqrt{3}\cos\phi)\mathbf{k}, \quad 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$

$= y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$

$\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$,
$\leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$

and Examples

: C be the smooth curve $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + - 2\cos^3 t)\mathbf{k}$, oriented to be traversed counterclockwise around z-axis when viewed from above. Let S be the piecewise ooth cylindrical surface $x^2 + y^2 = 4$, below the curve for ≥ 0, together with the base disk in the xy-plane. Note that C ; on the cylinder S and above the xy-plane (see the accompany- figure). Verify Equation (4) in Stokes' Theorem for the vector ld $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.



rify Stokes' Theorem for the vector field $\mathbf{F} = 2xy\mathbf{i} + x\mathbf{j} + + z)\mathbf{k}$ and surface $z = 4 - x^2 - y^2, z \geq 0$, oriented with t normal $\mathbf{n}$ pointing upward.

21. Zero circulation Use Equation (8) and Stokes' Theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

a. $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ **b.** $\mathbf{F} = \nabla(xy^2z^3)$

c. $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ **d.** $\mathbf{F} = \nabla f$

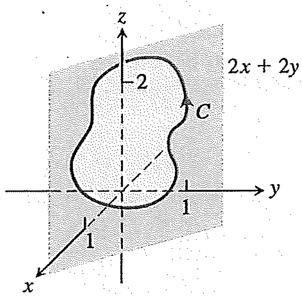
22. Zero circulation Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy-plane is zero

a. by taking $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, 0 \leq t \leq 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle.

b. by applying Stokes' Theorem.

23. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\oint_C 2y\,dx + 3z\,dy - x\,dz$$



depends only on the area of the region enclosed by C and not on the position or shape of C.

24. Show that if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \times \mathbf{F} = \mathbf{0}$.

25. Find a vector field with twice-differentiable components whose curl is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or prove that no such field exists.

26. Does Stokes' Theorem say anything special about circulation in a field whose curl is zero? Give reasons for your answer.

27. Let R be a region in the xy-plane that is bounded by a piecewise smooth simple closed curve C and suppose that the moments of inertia of R about the x- and y-axes are known to be I_x and I_y. Evaluate the integral

$$\oint_C \nabla(r^4) \cdot \mathbf{n}\,ds,$$

where $r = \sqrt{x^2 + y^2}$, in terms of I_x and I_y.

28. Zero curl, yet the field is not conservative Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)